The Negative Pell Equation and Fibonacci Sequences

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Abstract – Let N be a positive integer. We give a necessary and sufficient condition for the negative Pell equation $x^2 - Ny^2 = -1$ to be solvable, and show in this case, that all the solutions are expressible in terms of the fundamental solution of the Pell equation. The solutions of the Pell equation, as well as those of the negative Pell equation, if they exist, are shown to be expressible as generalized Fibonacci sequences. The solubility of the equation for different values of N is investigated.

Index Terms – Pell Equation, Fibonacci Sequences.

1. INTRODUCTION

The Pell Equation (PE(N) or, if N is not specified, PE) $x^2 - Ny^2 = 1$

and the negative Pell Equation (NPE(N) or NPE)

$x^2 - Ny^2 = -1$

have been studied since ancient times. Here N is a positive integer that is not an integral square, and the solutions sought are pairs (x,y) of integers.

A positive solution (x,y) is one where both x and y are positive. Clearly all solutions are known if the positive ones are known. For $N = 1$, $(\pm 1, 0)$ (resp. $(0, \pm 1)$ are the only solutions of PE (resp. NPE). Since two consecutive numbers cannot both be perfect squares, no solutions exist for either PE or NPE for square numbers $N > 1$.

In what follows N will be a positive integer that is not a square.

PE is always solvable. Bhaskaracharya – also known as Bhaskar II - used the method of "cyclics" proposed by Brahmagupta in the 7th century, to determine a solution of PE(N) for natural numbers $N > 1$. Euler used continued fractions to find solutions, but did not confirm existence of solutions. Lagrange (1768) took Euler's work further, and solved the problem completely, by proving that a solution always exists and then finding all solutions from the fundamental solution.

Theorem 1 (Lagrange):

Let y_1 be the smallest positive integer having the property that $x_1 = \sqrt{Ny_1^{\,2} + 1}$ is a positive integer. The pair (x_1, y_1) is called the *fundamental solution* of PE. The fundamental solution can be obtained by writing $\sqrt{N}$ as a continued fraction. Let

$(x_1 + y_1\sqrt{N})^n = x_n + y_n\sqrt{N}$, $n = 1, 2, \ldots$

.Then $\{(x_n, y_n)\}_{n=1,2..}$ is the set of all positive solutions.

NPE on the other hand is not always solvable. For example, if $N \equiv 3 \bmod 4$, $x^2 = (Ny^2 - 1) \equiv_4 2 \text{ or } 3$ has no solution.

The set $Z[\sqrt{N}] = \{x + y\sqrt{N} \mid x \in \mathbf{Z},\ y \in \mathbf{Z}\}$ is additive subgroup of the reals, generated by the two element set $\{1, \sqrt{N}\}$. Clearly $(x;y) \in \mathbf{Z}^2$ is a solution of PE (resp, NPE) , where $\alpha = x + y\sqrt{N}$. $\Leftrightarrow \alpha\bar{\alpha} = 1$ (resp.$\alpha\bar{\alpha} = -1$) . Also $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$, whenever α & β are in $\mathbf{Z}[\sqrt{N}]$ as can be easily verified.

2. THE FIBONACCI CONNECTION

We now show that the solutions of PE and NPE, when they exist, are representable as generalised Fibonacci sequences.

Definition: We say that a sequence $< a_n >$, $n = 1, 2, 3,\ldots$ of non-negative integers is a *generalised Fibonacci sequence* (GFS) *of type* (r, s) whenever there exists integers r and s such that for $n = 3, 4, 5,\ldots$,

$a_n = ra_{n-1} + sa_{n-2}$.

Clearly such a sequence is completely known if its first two terms and its type are known.

Theorem 2:

Suppose $x^2 - Ny^2 = -1$ is solvable. Let (X_1, Y_1) be its fundamental solution, that is, Y_1 is the smallest positive integer such that $X_1 = \sqrt{NY_1^2 - 1}$ is a positive integer. Let

$\sigma = X_1 + Y_1\sqrt{N}$ and $\sigma^n = X_n + Y_n\sqrt{N}$, $n = 1, 2, 3,\ldots$

Clearly for each n, X_n and Y_n are positive integers.

Then

(1) $\sigma^2 = 2X_1\sigma + 1$

(2) For each $n = 2, 3,\ldots$

$$X_{n+1} = 2X_1X_n + X_{n-1}$$
$$Y_{n+1} = 2X_1Y_n + Y_{n-1}$$

(3) If $\rho=\sigma^2=X_2+Y_2\sqrt{N}$, then (X_2,Y_2) is the fundamental solution of PE(*N*). Hence the sequence $<\sigma^n>$ is a GFS of type $(2X_1,1)$, having first two terms, the fundamental solutions of NPE(*N*) and PE(*N*) respectively.

(4) The positive solution sets of PE and NPE are, respectively, $\{(X_{2n},Y_{2n})\}_{n=1,2,3,...}$ and $\{(X_{2n-1},Y_{2n-1})\}_{n=1,2,3,...}$

(5) If $x_1+y_1\sqrt{N}$ is the fundamental solution of PE(*N*), then $x_1=X_2=2X_1^2+1$

Proofs:

(1) $2X_1\sigma+1=2X_1^2+1+2X_1Y_1\sqrt{N}=X_1^2+2X_1Y_1\sqrt{N}+NY_1^2=\sigma^2$

$$\sigma^2=2X_1\sigma+1$$
$$\Rightarrow \forall n\geq 2,\ \sigma^{n+1}=2X_1\sigma^n+\sigma^{n-1}$$
$$\Rightarrow \forall n\geq 2,\ X_{n+1}+Y_{n+1}\sqrt{N}=2X_1(X_n+Y_n\sqrt{N})+X_{n-1}+Y_{n-1}\sqrt{N}$$

(2) $\Rightarrow \forall n\geq 2,\ X_{n+1}=2X_1X_n+X_{n-1}$ and $Y_{n+1}=2X_1Y_n+Y_{n-1}$

(3) Let $\rho=\sigma^2=(X_1+Y_1\sqrt{N})^2$. It is known that ρ is the fundamental solution to the Pell equation. (see for example, page 356 of [1]).

(4) Hence $\{(\rho^n)^*\}=\{(\sigma^{2n})^*\}=\{(X_{2n},Y_{2n})\}_{n=1,2,3}$ is the set of positive solutions of PE.

For any *n* = 1,2,3,...

$$\begin{aligned} X_{2n-1}^2-Ny_{2n-1}^2 &= (X_{2n-1}-Y_{2n-1}\sqrt{N})(X_{2n-1}+Y_{2n-1}\sqrt{N}) \\ &= \overline{\sigma^{2n-1}}\sigma^{2n-1} \\ &= (-1)^{2n-1} \\ &= -1 \end{aligned}$$

so (X_{2n-1},Y_{2n-1}) is a solution of NPE for all positive integers *n*. It remains to show that there are no other positive solutions to NPE. Let $\alpha^*=(a,b)$ be any positive solution of NPE. Then α^2 is a positive solution of PE and hence there exists *m* such that $\alpha^2=\sigma^{2m}$. Since both α and σ^m are positive we have $\alpha=\sigma^m$. Now *m* cannot be even, for then α would be a solution of NPE as well as PE, which is impossible.

(5) is immediate.

Next we show that the solution sets of PE(*N)* and NPE(*N*) are each generalised Fibonacci sequences.

Theorem 3:

(1) Let *N* be a positive integer that is not a square number. Let $\rho=x_1+y_1\sqrt{N}$ be the fundamental solution PE(*N*): $x^2-Ny^2=1$. Let $(x_1+y_1\sqrt{N})^n=x_n+y_n\sqrt{N}$. Then each of $<x_n>$ and $<y_n>$ is a GFS of type $(2x_1,-1)$.

(2) Suppose NPE is solvable with fundamental solution $\sigma=X_1+Y_1\sqrt{N}$. Then each of the sequences $<X_{2n-1}>$ and $<Y_{2n-1}>$ is a GFS of type $(2X_2,-1)=(4X_1^2+2,-1)=(2x_1,-1)$ where $x_1+y_1\sqrt{N}$ is the fundamental solution of the PE(N). The first two solutions of NPE(*N*) are (X_1,Y_1) and $(4X_1^3+3X_1,(4X_1^2+1)Y_1)$.

Proofs:

(1) For any *n* = 2, 3, 4, 5...

$$\begin{aligned} &(2x_1x_n-x_{n-1})+(2x_1y_n-y_{n-1})\sqrt{N} \\ &=2x_1\rho^n-\rho^{n-1} \\ &=\rho^{n-1}[2x_1(x_1+y_1\sqrt{N})-1] \\ &=\rho^{n-1}[x_1^2+2x_1y_1\sqrt{N}+Ny_1^2] \\ &=\rho^{n-1}\rho^2 \\ &=x_{n+1}+y_{n+1}\sqrt{N} \end{aligned}$$

Since $\sqrt{N}$ is irrational,

$$\begin{aligned} (x_{n+1},y_{n+1}) &= (2x_1x_n-x_{n-1},2x_1y_n-y_{n-1)}\sqrt{n}) \\ &= 2x_1(x_n,y_n)-(x_{n-1},y_{n-1})\sqrt{N} \end{aligned}$$

proving (1).

(2) Recall that $\sigma^2=2X_1\sigma+1$. Hence

$$\sigma^3=2X_1\sigma^2+\sigma=(4X_1^2+1)\sigma+2X_1$$

$$\begin{aligned} \sigma^{2n+1} &= \sigma^{2n-2}\sigma^3 \\ &=(4X_1^2+1)\sigma^{2n-1}+2X_1\sigma^{2n-2} \\ &=(4X_1^2+1)\sigma^{2n-1}+\sigma^{2n-3}(2X_1\sigma) \\ &=(4X_1^2+1)\sigma^{2n-1}+\sigma^{2n-3}(\sigma^2-1) \\ &=2(2X_1^2+1)\sigma^{2n-1}-\sigma^{2n-3} \\ &=2X_2\sigma^{2n-1}-\sigma^{2n-3} \end{aligned}$$

$$\begin{aligned} \sigma^3 &=(4X_1^2+1)\sigma+2X_1 \\ &=(4X_1^2+1)(X_1+Y_1\sqrt{N})+2X_1 \\ &=(4X_1^3+3X_1)+(4X_1^2+1)Y_1\sqrt{N} \end{aligned}$$

the result follows.

3. THE NEGATIVE PELL EQUATION

We now investigate some values of *N* for which NPE(N) is solvable/unsolvable.

Theorem 4:

Let $\rho = x_1 + y_1\sqrt{N}$ be the fundamental solution of the Pell Equation $x^2 - Ny^2 = 1$. Then the negative Pell equation $x^2 - Ny^2 = -1$ is solvable if and only if $X_1 = \sqrt{\frac{x_1 - 1}{2}}$ and $Y_1 = \frac{y_1}{2X_1}$ are both integers.

Proof : We have seen that if NPE is solvable with fundamental solution $\sigma = X_1 + Y_1\sqrt{N}$, then

$(X_1 + Y_1\sqrt{N})^2 = x_1 + y_1\sqrt{N}$. And then $x_1 = 2X_1^2 + 1$ and $y_1 = 2X_1Y_1$. Since X_1 and Y_1 are positive integers, sufficiency is established. For the converse, note that

$(2X_1^2 + 1)^2 = x_1^2 = Ny_1^2 + 1 = 4NX_1^2Y_1^2 + 1$ from which it easily follows that (X_1, Y_1) is a solution to NPE(N).

Theorem 5:

(1) Let N be a prime $\equiv 1 \bmod 4$. Then NPE(N) is solvable.

(2) For any integer k, NPE($k^2 + 1$) is solvable. The first two solutions of NPE($k^2 + 1$) are $(k, 1)$ and $(4k^3 + 3k, 4k^2 + 1)$ and its type is $(4k^2 + 2, -1)$.

(3) If NPE(N) is solvable then $N = 2^{\varepsilon}\prod_i p_i$ where $\varepsilon = 0$ or 1 and each p_i is a prime $\equiv 1 \bmod 4$. In particular, if NPE(N) is solvable, then $N \equiv (1 \text{ or } 2) \bmod 4$.

Proofs:

(1) See page 357#12 in [1].

(2) Since $k^2 - (k^2 + 1)1^2 = -1$, NPE$(k^2 + 1)$ is solvable,

$(X_1, Y_1) = (k, 1)$ is the fundamental solution of NPE $(1 + k^2)$. It is easy to see that the next solution, namely, (X_3, Y_3) is $(4k^3 + 3k, 4k^2 + 1)$. Its type is $(2X_2, -1) = (2(2k^2 + 1), -1) = (4N - 2, -1)$.

(3) Let (x, y) be a solution of NPE(N). Then $Ny^2 = x^2 + 1$ is a sum of two squares and hence of the form $Ny^2 = 2^{\in} m^{2j} n$, where $\in$ is a non-negative integer, m is a product of primes $\equiv 3 \bmod 4$ and n is a product of (not necessarily distinct) primes $\equiv 1 \bmod 4$.(See Theorem 2.15 of [1])

If x is even, then Ny^2 is odd and $\in = 0$. If x is odd, then Ny^2 is of the form 2(odd) in which case $\in = 1$. So $\in = 0$ or 1.

Suppose p is a prime factor of N. Then $x^2 + 1 = Ny^2 \equiv_p 0$. By Theorem 2.12 of [1], $p = 2$ or $p \equiv 1 \bmod 4$ Hence $x^2 + 1 = Ny^2 = 2^{\varepsilon}\prod_i p_i$ where $\varepsilon = 0$ or 1, and each p_i is a prime $\equiv 1 \bmod 4$. Clearly, then, N, too, must be of this form, and then $N \equiv (1 \text{ or } 2) \bmod 4$.

Example : Let k = 4 in Theorem 5(2). The theorem says that (4;1) and (268;65) are the fundamental solutions of NPE(17) and PE(17) resp. The type is (66;-1) so the rest of the positive solutions of NPE(17) are immediate, the first two being

(4;1) and 66(268; 65) - (4; 1)=(17684; 4289)

Theorem 6:

(1) For any integer k >0, NPE $(k^2 + 2)$ is not solvable.

(2) For any integer k >1, NPE $(k^2 - 1)$ is not solvable.

(3) For any integer k > 2, NPE $(k^2 - 2)$ is not solvable.

For example, NPE(194)=NPE($14^2 - 2$) is not solvable.

Hence for $k > 2$, of the five consecutive integers $k^2 - 2$, $k^2 - 1$, k^2, $k^2 + 1$, $k^2 + 2$, only NPE $(k^2 + 1)$

Proofs:

(1) From $(k^2 + 1)^2 - (k^2 + 2)k^2 = 1$ we see that $(k^2 + 1, k)$ is solution to PE $(k^2 + 2)$, and it is minimal, so its fundamental solution is $(x_1, y_1) = (k^2 + 1, k)$. If NPE $(k^2 + 2)$ is solvable then $\frac{k^2 + 1 - 2}{2} = \frac{k^2}{2}$ is a perfect square, a contradiction.

(2) Let $k > 1$ be an integer, and let $N = k^2 - 1$. Then

$N \equiv (0 \text{ or } 3) \bmod 4$, so from Theorem 5(3), NPE $(k^2 - 1)$ is not solvable.

(3) $(k^2 - 1)^2 - (k^2 - 2)k^2 = 1$ so $(k^2 - 1, k)$ is solution to PE $(k^2 - 2)$, and it is minimal, so its fundamental solution is $(x_1, y_1) = (k^2 - 1, k)$. If NPE $(k^2 - 2)$ is solvable with fundamental solution (X_1, Y_1), then from Theorem 4, $\frac{y_1}{2X_1} = \frac{k}{2\sqrt{\frac{k^2 - 2}{2}}} = \frac{k}{\sqrt{2k^2 - 4}}$ is an integer. However, the expression on the right is < 1 whenever $k > 2$, a contradiction.

The results above identify the following integers N between 1 and 100 for which NPE(N) is solvable:

2, 5, 10, 13, 17, 26, 29, 37, 41, 50, 53, **58,** 61, 65, 73, **74,** 82, **85,** 89, 97

This is a complete list. The ones marked in bold were identified using the fundamental solutions of PE(*N*) and Theorem 4. The presence of consecutive integers in this list could pose a challenge to those who seek a generic solution to identify all *N* for which NPE(*N*) is solvable.

4. ALGEBRAIC CONSIDERATIONS

Suppose NPE(*N*) is solvable with fundamental solution $\sigma = X_1 + Y_1\sqrt{N}$ (Strictly speaking, $(X_1;Y_1)$ is a solution, not σ).

A theorem of Dirichlet (1805-1859) from algebraic number theory, describes the structure of the group of units of a general ring of algebraic integers, of which the ring $\mathbf{Z}[\sqrt{N}]$ is an example. It can be used to conclude that the units of the ring $\mathbf{Z}[\sqrt{N}]$, namely its multiplicative group of invertible elements, is the product of {±1} and an infinite cyclic group. The invertible elements of $\mathbf{Z}[\sqrt{N}]$ are precisely those elements that have norm 1 or – 1, where the norm of $x + y\sqrt{N}$ in $\mathbf{Z}[\sqrt{N}]$ Is defined as $x^2 - Ny^2$. But this set is precisely the totality of solutions of PE(*N*) and NPE(*N*). The generator of the cyclic group is just the fundamental solution of NPE(*N*), namely, σ. The totality of solutions to PE and NPE is the multiplicative subgroup $G = <-1, \sigma>$ of the positive reals generated by -1 and σ. (See [2]).The solutions to PE(*N*) is the subgroup $H < -1, \sigma^2 >$ generated by -1 and σ^2, while the coset σH identifies all the solutions of NPE(*N*):

$(x; y)$ is a positive solution of NPE(*N*) $\Leftrightarrow x + y\sqrt{N} \in < \sigma^2 >$

$(x; y)$ is a positive solution of PE(*N*) $\Leftrightarrow x + y\sqrt{N} \in \sigma < \sigma^2 >$

Note that the mapping $x + y\sqrt{N} \rightarrow \begin{pmatrix} x & y \\ Ny & x \end{pmatrix}$ establishes a ring isomorphism between $\mathbf{Z}[\sqrt{N}]$ and a subring of $M_2(\mathbf{Z})$, the ring of 2x2 matrices over the integers.

The graphs $\{(x, y) \in \mathbf{R}^2 \mid x^2 - Ny^2 = \pm 1\}$ comprise four non-intersecting hyperbolic pieces, asymptotic with the lines having equations $y = \pm \frac{x}{\sqrt{N}}$. The totality of solutions to PE and NPE are the integral lattice points on these curves.

REFERENCES

[1] Niven, Zuckerman & Montgomery, *An Introduction to the Theory of Numbers*, John Wiley & Sons, 1991 (5th Edition)

[2] H.W. Lenstra Jr., Solving the Pell Equation, *Notices of AMS* (2002), Vol 49, No.2.